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Gravitational Collapse in Loop Quantum Gravity

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Abstract In this paper we study the gravitational collapse applying methods of loop quantum gravity to a minisuperspace model. We consider the space-time region inside the Schwarzschild black hole event horizon and we divide this region in two parts, the first one where the matter (dust matter) is localized and the other (outside) where the metric is Kantowski–Sachs type. We study the Hamiltonian constraint obtaining a set of three difference equations that give a regular and natural evolution beyond the classical singularity point in "$r = 0$" localized.

1 Introduction

Quantum gravity, the theory that wants to reconcile general relativity and quantum mechanics, is one of the major problems in theoretical physics today. The lesson of general relativity is that also the space-time is dynamical, then it is not possible to study the other interaction on a fixed background. The background itself is a dynamical field.

One of the more widespread theory of quantum gravity is the theory called of "loop quantum gravity" [1–5]. This is one of the nonperturbative and background independent approach to quantum gravity (another nonperturbative approach to quantum gravity is called "asymptotic safety quantum gravity" [6]).

In this paper following [7–9] we apply ideas suggested by full loop quantum gravity to a particular minisuperspace model that we obtain imposing symmetries on the full metric. In our model as in [7–9] it is possible to implement the Dirac quantization program following the fundamental ideas of loop quantum gravity.

Some interesting results in this theory are related to the problem of space-like singularity. In fact it was shown in [10–15] that it is possible to solve the cosmological singularity problem and the black hole singularity problem by using the ideas developed in full loop quantum gravity theory. In [16, 17] the problem of black hole singularity has been analyzed also under the contest of "asymptotic safety quantum gravity"; in that paper authors showed

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that nonperturbative quantum gravity effects give a much less singular Schwarzschild metric.

In this paper we study the gravitational collapse of a dust sphere inside the event horizon following the paper [18]. We consider the phase of the gravitational collapse when all (dust) matter has crossed the Schwarzschild radius. In this phase the symmetric reduced metric is homogeneous and it is possible to study the quantum Einstein equations for the minisuperspace model. In particular we have two regions (Region 1 and Region 2), the first one is where the matter is localized (Region 1) with a Friedmann–Robertson–Walker type metric and the other one outside the matter (Region 2) where the metric is of Kantowski–Sachs type [19]. We summarize the model in ADM variables following the paper of Nambu and Sasaki [18]. After this we will pass to Ashtekar variables. In Ashtekar variables we quantize the system following ideas suggested from full loop quantum gravity [1–5]. The technology used in this paper was developed in the preview papers [13–15, 20, 21, 23].

The paper is organized as follow. In Sect. 2 we briefly recall properties of the gravitational collapse in ADM variables inside the event horizon, $r < 2MG_N$, [18]. As well known, here the temporal and spatial (radial) coordinates exchange their role and so it is possible to study an homogeneous space-time. In Sect. 3 we recall the Ashtekar formulation of general relativity in terms of the gauge connection A_a^i and of the density triad E_i^a. In particular we recall the form of the Ashtekar connection and of the density triad inside the event horizon. For the symmetric reduced connection we introduce the holonomy and we define the classical Hamiltonian constraint in terms of holonomies and of the volume operator. At the end of this section we define the boundary conditions on the S^2 separation sphere (between Region 1 and Region 2 of above) in terms of holonomies. In Sect. 4 we recall the polymer quantization scheme and we introduce the kinematical Hilbert space for the gravitational collapse. In Sect. 5 we study the dynamics identifying the degrees of freedom and deriving a set of difference equations that give a regular and natural evolution beyond the classical singularity [34–36]. Throughout the paper we use the units: $c = 1$ and $\kappa^2 = 8\pi G_N$.

2 Gravitational Collapse in ADM Variables Inside a Black Hole (Classical Theory)

In this section we recall the results of [18] in ADM variables. We consider the space-time region inside the black hole horizon and we study the collapse of a dust sphere. The metric inside the matter for a co-moving observer is described by a closed Friedmann universe which is homogeneous and isotropic,

$$ds_1^2 = -N_1^2(t)dt^2 + R^2(t)(d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2)) \quad (0 \leqslant \chi \leqslant \chi_0). \tag{1}$$

The motion of dust matter is given by: $R(\eta) = R_0(1+\cos\eta)/2$ and $t(\eta) = R_0(\eta + \sin\eta)/2$, $(0 \leqslant \eta \leqslant \pi)$. Where R_0 and χ_0 are related to the initial radius R_i of dust sphere in Schwarzschild radial coordinate and to Schwarzschild mass M_{Schw} by: $R_i = R_0\sin\chi_0$, $M_{\text{Schw}} = R_0\sin^3\chi_0/2$. In Fig. 1 and Fig. 2 is illustrated the collapse. We define Region 1 the region inside the matter and Region 2 the exterior. The metric in Region 2 is the Schwarzschild metric. Hence, if we use co-moving coordinates in Region 1 and Schwarzschild coordinates in Region 2, we obtain that the hypersurfaces becomes homogeneous in any region inside the event horizon. The homogeneous metric outside the collapsing star is of Kantowski–Sachs type [19]

$$ds_2^2 = -N_2^2(t)dt^2 + a^2(t)d\chi^2 + b^2(t)(d\theta^2 + \sin^2\theta d\phi^2) \quad (\chi_0 \leqslant \chi < \infty). \tag{2}$$

Fig. 1 Spatial section for the gravitational collapse inside the event horizon for the particular value of the radial coordinate $\chi_0 = \pi/2$

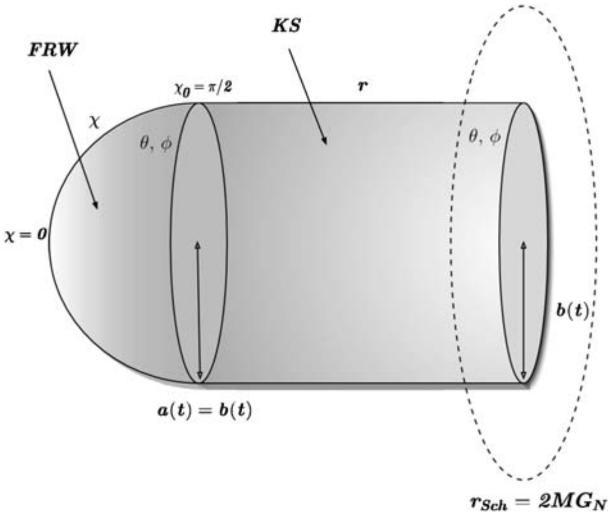
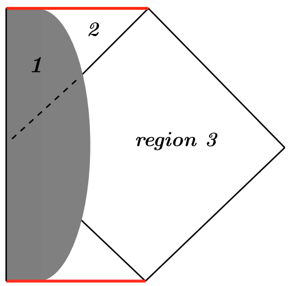


Fig. 2 Penrose diagram for the gravitational collapse inside the event horizon (Region 1 and Region 2) and outside the event horizon (Region 3). This is a schematic causal diagram of expansion and collapse of a dust sphere. Region 1 is described by a Friedmann–Robertson–Walker metric and Region 2 by a Kantowski–Sachs metric. Region 3 is described by the external event horizon Schwarzschild metric



We have assumed the 3-surface is described by the same radial coordinate χ also in the exterior of the dust matter region.

The action in ADM variables using (1) and (2) is

$$
\begin{aligned}
S &= \frac{1}{2\kappa}\int d^3x N\sqrt{h}(K^a_b K^a_b - (K^a_a)^2 + {}^{(3)}\mathrm{R} - 2\kappa\rho_{\mathrm{dust}}) \\
&= \frac{\mathrm{V}_1 R^3}{2\kappa}\left[-\frac{6}{N_1}\left(\frac{\dot R}{R}\right)^2 + N_1\left(\frac{6}{R^2} - 2\kappa\rho_{\mathrm{dust}}\right)\right] \\
&\quad + \frac{V_2 ab^2}{2\kappa}\left[-\frac{1}{N_2}\left(2\left(\frac{\dot b}{b}\right)^2 + 4\frac{\dot a\dot b}{ab}\right) + \frac{2N_2}{b^2}\right],
\end{aligned} \tag{3}
$$

where $\mathrm{V}_1 = 4\pi\int_0^{\chi_0} d\chi \sin^2\chi$, $\mathrm{V}_2 \equiv 4\pi\mathcal{L}\int_{\chi_0}^{\chi_1} d\chi \equiv 4\pi\mathcal{L}(\chi_1 - \chi_0)$ and $\kappa = 8\pi G_N$ (the index $a, b = 1, 2, 3$). I introduced $\mathcal{L}$ for dimensional reason ($[\mathcal{L}] = L$) in V_1, and χ_1 is a cut-off on the space radial coordinate. The spatial homogeneity enable us to fix a linear radial cell and restrict all integrations to this cell [10, 11]. To simplify notations we restrict the linear radial cell to $\chi_1 - \chi_0 = 1$. In (3) following [18] we have not considered the ambiguous term $\int N\sqrt{h}^{(3)}R$ at the junction point $\chi = \chi_0$.

The Hamiltonian corresponding to the action (3) is

$$H = p_R\dot{R} + p_a\dot{a} + p_b\dot{b} - L$$
$$= N_1\left(-\frac{\kappa}{12\mathrm{V}_1}\frac{p_R^2}{R} - \frac{3\mathrm{V}_1}{\kappa}R + M_{\mathrm{dust}}\right) + N_2\left[\frac{2\kappa}{\mathrm{V}_2}\left(-\frac{p_a p_b}{4b} + \frac{a p_a^2}{8b^2}\right) - \frac{\mathrm{V}_2}{\kappa}a\right], \quad (4)$$

where $M_{\mathrm{dust}} = \mathrm{V}_1 R^3 \rho_{\mathrm{dust}}$ is the constant total dust matter and the momentum conjugate to the 3-metric q_{ab} is

$$\Pi^a_b = -\frac{1}{2\kappa}(K^a_b - K h^a_b) = \begin{cases} -\frac{1}{2\kappa N_1}\sin\theta\,\mathrm{diag}\big(2\dot{R}\sin^2\chi, 2\dot{R}, 2\frac{\dot{R}}{\sin^2\theta}\big) & \text{if } \chi < \chi_0; \\ -\frac{1}{2\kappa N_2}\sin\theta\,\mathrm{diag}\big(\frac{2b\dot{b}}{a}, \dot{a} + \frac{a\dot{b}}{b}, \frac{a\dot{b}}{b\sin^2\theta}\big) & \text{if } \chi > \chi_0. \end{cases} \quad (5)$$

The momentum conjugate to the variables R, a, b are

$$p_R = -\frac{6\mathrm{V}_1}{\kappa N_1}R\dot{R}, \qquad p_a = -\frac{2\mathrm{V}_2}{\kappa N_2}b\dot{b}, \qquad p_b = -\frac{2\mathrm{V}_2}{\kappa N_2}(a\dot{b} + b\dot{a}). \quad (6)$$

To obtain a correlation between Region 1 and Region 2 we have to take into account the momentum constraint ($H^i \sim \Pi^{ij}_{|j} = 0$) on the S^2-sphere junction surface that is in $\chi = \chi_0$ localized. The θ and ϕ component of the momentum constraint are trivial because the spherical symmetry. The χ component of the momentum constraint is identically zero for $\chi \neq \chi_0$, but when we impose the condition $\Pi^{\chi j}_{|j} = 0$ for $\chi = \chi_0$ we obtain the following unambiguous junction conditions [18]

$$\begin{aligned} P_1 &= \frac{p_R \sin^2\chi_0}{3\mathrm{V}_1} - \frac{p_a}{\mathrm{V}_2} = 0, \\ P_2 &= R\sin\chi_0 - b = 0. \end{aligned} \quad (7)$$

We can express the first of the relations (7) in terms of $R, \dot{R}, b, \dot{b}$ and we obtain the new constraints set

$$\begin{aligned} P_1 &= \frac{R\dot{R}\sin^2\chi_0}{N_1} - \frac{b\dot{b}}{N_2} = 0, \\ P_2 &= R\sin\chi_0 - b = 0. \end{aligned} \quad (8)$$

The gravitational collapse inside the horizon in ADM variables is completely defined by the following four constraints [18]

$$\begin{aligned} H_1 &= -\frac{\kappa}{12\mathrm{V}_1}\frac{p_R^2}{R} - \frac{3\mathrm{V}_1}{\kappa}R + M_{\mathrm{dust}} = 0, \\ H_2 &= \frac{2\kappa}{\mathrm{V}_2}\left(-\frac{p_a p_b}{4b} + \frac{a p_a^2}{8b^2}\right) - \frac{\mathrm{V}_2}{\kappa}a = 0, \\ P_1 &= \frac{p_R \sin^2\chi_0}{3\mathrm{V}_1} - \frac{p_a}{\mathrm{V}_2} = 0, \\ P_2 &= R\sin\chi_0 - b = 0. \end{aligned} \quad (9)$$

Solving the constraints equations (9) we obtain the known results for the classical dust matter gravitational collapse [18].

3 Gravitational Collapse in Ashtekar Variables

In this section we study the gravitational collapse in Ashtekar variables [22]. In particular we will express the Hamiltonian constraint inside and outside the matter and the constraints P_1 and P_2 in terms of the symmetric reduced Ashtekar connection [23–28].

3.1 Ashtekar variables

In LQG the fundamental variables are the Ashtekar variables: they consist of an $SU(2)$ connection A_a^i and the electric field E_i^a, where $a, b, c, \ldots = 1, 2, 3$ are tensorial indices on the spatial section and $i, j, k, \ldots = 1, 2, 3$ are indices in the $su(2)$ algebra. The density weighted triad E_i^a is related to the triad e_a^i by the relation $E_i^a = \frac{1}{2}\epsilon^{abc}\epsilon_{ijk}e_b^j e_c^k$. The metric is related to the triad by $q_{ab} = e_a^i e_b^j \delta_{ij}$. Equivalently,

$$\sqrt{\det(q)}q^{ab} = E_i^a E_j^b \delta^{ij}. \tag{10}$$

The rest of the relation between the variables (A_a^i, E_i^a) and the ADM variables (q_{ab}, K_{ab}) is given by

$$A_a^i = \Gamma_a^i + \gamma K_{ab} E_j^b \delta^{ij}, \tag{11}$$

where γ is the Immirzi parameter and Γ_a^i is the spin connection of the triad, namely the solution of Cartan's equation: $\partial_{[a}e_{b]}^i + \epsilon_{jk}^i \Gamma_{[a}^j e_{b]}^k = 0$.

The action is

$$S = \frac{1}{\kappa\gamma}\int dt \int_\Sigma d^3x[-2\,\mathrm{Tr}(E^a \dot{A}_a) - N\mathcal{H} - N^a\mathcal{H}_a - N^i\mathcal{G}_i], \tag{12}$$

where N^a is the shift vector, N is the lapse function and N^i is the Lagrange multiplier for the Gauss constraint $\mathcal{G}_i$. We have introduced also the notation $E_{[1]} = E^a\partial_a = E_i^a\tau^i\partial_a$ and $A_{[1]} = A_a dx^a = A_a^i\tau^i dx^a$. The functions $\mathcal{H}$, $\mathcal{H}_a$ and $\mathcal{G}_i$ are respectively the Hamiltonian, diffeomorphism and Gauss constraints, given by

$$\begin{aligned}
&\mathcal{H}(E_i^a, A_a^i) = -4e^{-1}\,\mathrm{Tr}(F_{ab}E^aE^b) - 2e^{-1}(1+\gamma^2)E_i^a E_j^b K_{[a}^i K_{b]}^j, \\
&\mathcal{H}_b(E_i^a, A_a^i) = E_j^a F_{ab}^j - (1+\gamma^2)K_a^i G_i, \\
&\mathcal{G}_i(E_i^a, A_a^i) = \partial_a E_i^a + \epsilon_{ij}^k A_a^j E_k^a,
\end{aligned} \tag{13}$$

where the curvature field strength is $F_{ab} = \partial_a A_b - \partial_b A_a + [A_a, A_b]$ and the determinant is $e = \det(e_a^i) \equiv \sqrt{|\det(E_{[1]})}\mathrm{sgn}(\det(E_{[1]})$ [12].

In the rest of the paper we will consider only the first term of the Hamiltonian constraint to simplify the model. The second term will be essential to study a realistic gravitational collapse with real Immirzi parameter γ.

3.2 The Hamiltonian Constraint Inside the Matter

In this section we recall the Ashtekar variables for an homogeneous and isotropic space-time of topology $R \times S^3$ [23]. For this space the Ashtekar's connection and the densitized triad are

$$A_a^i = c\omega_a^I \delta_I^i, \qquad E_i^a = pX_I^a \delta_i^I, \tag{14}$$

where ω^I are the left-invariant one-forms and X_I are the left-invariant vector fields fulfilling $\omega^I(X_J) = \delta^I_J$. The Hamiltonian constraint in terms of the variables (c, p) is

$$H = H_E^{(RW)} + H_M = \frac{12}{\kappa} c(1-c)\mathrm{sgn}(p)\sqrt{|p|} + H_M, \tag{15}$$

where $H_E^{(RW)}$ is the Euclidean part of the Hamiltonian constraint inside the matter and $H_M = M_{\text{dust}}$ is the Hamiltonian constraint for the dust matter. I recall that the metric of the space-time in the region where the matter is localized is

$$ds^2 = -dt^2 + a^2(t)(d\chi^2 + \sin^2\chi(\sin^2\theta d\phi^2 + d\theta^2)). \tag{16}$$

The volume operator for the space section inside the matter in Ashtekar variables is

$$V^{(RW)} = 2\pi(\chi_0 - \cos\chi_0 \sin\chi_0)|p|^{\frac{3}{2}} \equiv \mathrm{V}(\chi_0)|p|^{\frac{3}{2}}. \tag{17}$$

The classical theory is define by the symplectic structure $\{c, p\} = \gamma\kappa/3$ [23].

Holonomies and the Hamiltonian Constraint Inside the Matter We introduce the holonomies for a space-time homogeneous and isotropic in the direction I

$$h_I^{(RW)} = e^{\delta_0 c\tau_I} = \cos(c\delta_o/2) + 2\tau_I \sin(c\delta_0/2), \tag{18}$$

and we express the gravitational part of the Hamiltonian constraint in terms of the holonomies [23, 40]

$$H_E^{(RW)} = -\frac{8}{\kappa^2\gamma\delta_0^3 \mathrm{V}(\chi_0)} \times \sum_{IJK} \epsilon^{IJK}\,\mathrm{Tr}[h_I^{(RW)} h_J^{(RW)} h_I^{(RW)-1} h_J^{(RW)-1} h_{[I,J]}^{(RW)-1} h_K^{(RW)} \{h_K^{(RW)-1}, V^{(RW)}\}]. \tag{19}$$

3.3 The Hamiltonian Constraint Outside the Matter

We recall that outside the matter but inside the horizon we have a Kantowski–Sachs type space-time. An homogeneous but anisotropic space-time of spatial section Σ of topology $\Sigma \cong \mathbb{R} \times S^2$ is characterized by an invariant connection 1-form $A_{[1]}$ of the form [24–30]

$$\begin{aligned} A_{[1]} = {} & A(t)\tau_3 dr + (A_1(t)\tau_1 + A_2(t)\tau_2)d\theta + (A_1(t)\tau_2 - A_2(t)\tau_1)\sin\theta d\phi \\ & + \tau_3 \cos\theta d\phi. \end{aligned} \tag{20}$$

The τ_i are the generators of the $SU(2)$ fundamental representation. They are related to the Pauli σ_i matrix by $\tau_i = -\frac{i}{2}\sigma_i$. On the other side the dual invariant densitized triad is

$$E_{[1]} = E(t)\tau_3 \sin\theta \frac{\partial}{\partial r} + (E^1(t)\tau_1 + E^2(t)\tau_2)\sin\theta\frac{\partial}{\partial\theta} + (E^1(t)\tau_2 - E^2(t)\tau_1)\frac{\partial}{\partial\phi}. \tag{21}$$

The coordinate r is related to the coordinate χ used in the first section by $r = l_P\chi$.

We are interested to the Kantowski–Sachs space-time with space section of topology $\mathbb{R} \times S^2$; the connection $A_{[1]}$ is more simple than in (20) with $A_2 = A_1$, and in the triad (21) we can choose the gauge $E^2 = E^1$ [37]. There is a residual gauge freedom on the pair (A_1, E^1). This is a discrete transformation $P : (A_1, E^1) \to (-A_1, -E^1)$; we have to fix this symmetry on the Hilbert space. The Gauss constraint is automatically satisfied and the Euclidean part of the Hamiltonian constraint is

$$H_E = \frac{8\pi l_P \sqrt{2}\,\mathrm{sgn}(E)}{\sqrt{|E|}|E^1|}[2AEA_1E^1 + (2(A_1)^2 - 1)(E^1)^2]. \tag{22}$$

We recall that the relation between the metric and density triad formulation and that the volume of the space section Σ are

$$q_{ab} = \mathrm{diag}\left(\frac{2(E^1)^2}{|E|}, |E|, |E|\sin^2\theta\right), \qquad V = 4\pi\sqrt{2}l_P\sqrt{|E|}|E^1|. \tag{23}$$

The phase space consists of two canonical pairs A, E and A_1, E^1 and the simplectic structure is given by the Poisson brackets, $\{A, E\} = \frac{\kappa\gamma}{4\pi l_P}$ and $\{A_1, E^1\} = \frac{\kappa\gamma}{16\pi l_P}$ [14, 15]. The coordinates and the momenta have dimensions: $[A] = L^{-1}$, $[A_1] = L^0$, $[E] = L^2$ and $[E^1] = L$.

Holonomies and Hamiltonian Constraint Outside the Matter The elementary configuration variables used in LQG are given by the holonomies along curves in the spatial section $\mathbb{R} \times S^2$ and the fluxes of triads on a two-surface in $\mathbb{R} \times S^2$. Because of symmetry reduction, it is sufficient to consider a set of functions that is sufficiently large to separate points of the reduced phase space. We restrict our attention to three sets of curves. More precisely, we consider only spin networks [1–5] based on graph made just of radial edges, and of edges along circles in the θ-direction or at $\theta = \pi/2$. Considering graphs of this type, we can omit the term "$\tau_3 \cos\theta d\phi$" of the connection in the "ϕ" direction.

We introduce the fiducial triad ${}^o e^a_I = \mathrm{diag}(1, 1, \sin^{-1}\theta)$ and co-triad ${}^o\omega^I_a = \mathrm{diag}(1, 1, \sin\theta)$, and define the holonomy: $h = e^{\int A_{[1]}} = e^{\int A_{[1]a}dx^a} = e^{\int A^i_a \frac{dx^a}{d\lambda}\tau_i d\lambda} = e^{\int A^i_I {}^o\omega^I_a {}^o e^a_J u^J \tau_i d\lambda} = e^{\int A^i_I u^I \tau_i d\lambda}$, where $u^a = \frac{dx^a}{d\lambda} = (\frac{dr}{d\lambda}, \frac{d\theta}{d\lambda}, \frac{d\phi}{d\lambda})$ and $u^I = {}^o\omega^I_a u^a$. The holonomy along a curve in the direction "I" is given by

$$\begin{aligned} h_1 &= \exp\int A^i_1\tau_i dx^1 = \exp[A\mu_0 l_P \tau_3], \\ h_2 &= \exp\int A^i_2\tau_i dx^2 = \exp[A_1\mu_0(\tau_2 + \tau_1)], \\ h_3 &= \exp\int A^i_3\tau_i dx^3 = \exp[A_1\mu_0(\tau_2 - \tau_1)], \end{aligned} \tag{24}$$

where $A^i_1 = (0, 0, A)$, $A^i_2 = (A_1, A_1, 0)$ and $A^i_3 = (-A_1, A_1, 0)$. The connection in (24) is integrated in the direction "I"; $\mu_0 l_P$ is the length of the curve along the direction r, μ_0 is the length of the curve along the directions θ and ϕ [12].[1] The length are defined using

[1] Working without considering full LQG theory it comes out that μ_0 is a quantization ambiguity. But in full LQG there is a minimum (non zero) area eigenvalue and we can use this result to fix μ_0 and δ_0 (18). When we define the field curvature F_{ab} we consider the holonomy around a loop to determine the integral of F_{ab} on a surface of area $a_0 = \sqrt{3}\gamma/4l_P^2$. At this point we can fix the value of the parameters μ_0, δ_0 (removing the ambiguity) comparing the area spectrum in minisuperspace models to the area spectrum in full loop quantum gravity. Comparing the area minimum eigenvalue from LQG with the minimum eigenvalues in minisuperspace models we obtain $\mu_0 \sim \delta_0 \sim 1$ (in particular $\mu_0 \neq 0$ and $\delta_0 \neq 0$) [12, 14, 15].

the fiducial triad ${}^{o}e^{a}_{I}$. Introducing the normalized vectors $n^i_1 = (0, 0, 1)$, $n^i_2 = \frac{1}{\sqrt{2}}(1, 1, 0)$, $n^i_3 = \frac{1}{\sqrt{2}}(-1, 1, 0)$ we can rewrite the holonomy h_I in the direction "I" as

$$\begin{aligned} h_I &= \exp(\bar{A}_I n^i_I \tau_i) = \cos\left(\frac{\bar{A}_I}{2}\right) + 2n^i_I \tau_i \sin\left(\frac{\bar{A}_I}{2}\right), \\ h_1 &= \cos\left(\frac{A\mu_0 l_P}{2}\right) + 2\tau_3 \sin\left(\frac{A\mu_0 l_P}{2}\right), \\ h_2 &= \cos\left(\frac{\sqrt{2}A_1\mu_0}{2}\right) + \sqrt{2}(\tau_2 + \tau_1)\sin\left(\frac{\sqrt{2}A_1\mu_0}{2}\right), \\ h_3 &= \cos\left(\frac{\sqrt{2}A_1\mu_0}{2}\right) + \sqrt{2}(\tau_2 - \tau_1)\sin\left(\frac{\sqrt{2}A_1\mu_0}{2}\right), \end{aligned} \tag{25}$$

where $\bar{A}_{I=1} = Al_P\mu_0$ and $\bar{A}_{I=2} = \bar{A}_{I=3} = A_1\mu_0\sqrt{2}$.

We now write the Hamiltonian constraint (22) in terms of holonomies [40]

$$H_E = -\frac{16\pi}{\kappa\gamma\mu_0^3} \sum_{IJK} \epsilon^{IJK}\, \mathrm{Tr}[h_I h_J h_I^{-1} h_J^{-1} h_{[IJ]} h_K^{-1}\{h_K, V\}], \tag{26}$$

where $h_{[IJ]} = \exp(-\mu_0^2 C_{IJ}\tau_3) = \cos(\mu_0^2 C_{IJ}/2) - 2\tau_3 \sin(\mu_0^2 C_{IJ}/2)$ and $C_{IJ} = \delta_{2I}\delta_{3J} - \delta_{3_I}\delta_{2J}$.

3.4 Boundary Conditions in Ashtekar Variables

In this section we recall relations between the Ashtekar and the ADM variables. In Region 1 where the matter is localized the relations are [31]

$$c = \frac{1}{2}\left(1 - \gamma\frac{\dot{a}}{N_1}\right), \qquad |p| = a^2. \tag{27}$$

In Region 2 instead the relations can be obtained comparing the Hamiltonian constraint written in terms of ADM variables and in terms of Ashtekar variables. From this analysis we obtain

$$A = i\frac{\dot{a}}{N_2}, \qquad A_1 = \frac{i\dot{b}}{\sqrt{2}N_2}, \qquad |E| = b^2, \qquad (E^1)^2 = \frac{a^2b^2}{2}. \tag{28}$$

Introducing (27) and (28) in (8) obtaining

$$\sqrt{|p|}(2c-1)\sin^2\chi_0 - \sqrt{2|E|}A_1 = 0. \tag{29}$$

We can express (29) in terms of holonomies using (18) and (24) and we obtain

$$\frac{1}{\delta_0}\sqrt{|p|}[4\,\mathrm{Tr}(h_1^{(RW)}\tau_1) + \delta_0]\sin^2(\chi_0) - \frac{1}{\mu_0}\sqrt{2|E|}\,\mathrm{Tr}[h_2(\tau_2+\tau_1)] + O(\mu_0) = 0. \tag{30}$$

Introducing the explicit form of holonomies in (30) we obtain the following finally form for the boundary conditions (8)

$$\frac{1}{\delta_0}\sqrt{|p|}\left[4\sin\left(\frac{c\delta_0}{2}\right) + \delta_0\right]\sin^2(\chi_0) - \frac{1}{\mu_0}2\sqrt{2|E|}\sin\left(\frac{A_1\mu_0}{\sqrt{2}}\right) + O(\mu_0) = 0,$$

$$\sqrt{|p|}\sin(\chi_0) - \sqrt{|E|} = 0. \tag{31}$$

4 Quantum Kinematics

We want to quantize the collapse inside the horizon using techniques from loop quantum gravity. Now we build the kinematical Hilbert space $\mathcal{H}_{\text{kin}}$ [38]. We define a graph Γ as a countable number of triple of points $(c_i, \mu_{Ei}, \mu_{E^1 i})$, where $c_i, \mu_{Ei}, \mu_{E^1 i} \in \mathbb{R}$. We denote by Cyl_Γ the vector space of functions $f(c, A, A_1)$ $(f : \mathbb{R}^3 \to \mathbb{C})$ of the type

$$f(c, A, A_1) = \sum_{ij,k} f_{ijk} e^{\frac{i\mu_i c}{2} + \frac{i\mu_{Ej} l_P A}{2} + \frac{i\mu_{E^1 k} A_1}{\sqrt{2}}}, \tag{32}$$

where $c, A, A_1 \in \mathbb{R}$, $c_i, \mu_{Ej}, \mu_{E^1 k} \in \mathbb{R}$, $f_{ijk} \in \mathbb{C}$ and i, j, k run over a finite number of integers (labeling the points of the graph). We call the function $f(c, A, A_1)$ in Cyl_Γ cylindrical with respect to the graph Γ. We consider all possible graphs (the points and their number can vary from a graph to another) and denote by Cyl the infinite dimensional vector space of functions cylindrical with respect to some graph: $\text{Cyl} = \bigcup_\Gamma \text{Cyl}_\Gamma$. Thus, any element $f(c, A, A_1)$ of Cyl can be expanded as in (32), where the uncountable basis $e^{\frac{i\mu_i c}{2}} \otimes e^{\frac{i\mu_{Ei} l_P A}{2}} \otimes e^{\frac{i\mu_{E^1 j} A_1}{\sqrt{2}}}$ is now labeled by arbitrary real numbers (c, μ_E, μ_{E^1}). A basis in Cyl is given by $|\mu, \mu_E, \mu_{E^1}\rangle \equiv |\mu\rangle \otimes |\mu_E\rangle \otimes |\mu_{E^1}\rangle$. Introducing the standard bra-ket notation we can define a basis [12] in the Hilbert space via

$$\langle c|\mu\rangle \otimes \langle A|\mu_E\rangle \otimes \langle A_1|\mu_{E^1}\rangle = e^{\frac{i\mu c}{2}} \otimes e^{\frac{i\mu_E l_P A}{2}} \otimes e^{\frac{i\mu_{E^1} A_1}{\sqrt{2}}}. \tag{33}$$

The basis states (33) are normalizable in contrast to the standard quantum mechanical representation and they satisfy

$$\langle \mu, \mu_E, \mu_{E^1}|\nu, \nu_E, \nu_{E^1}\rangle = \delta_{\mu,\nu}\delta_{\mu_E,\nu_E}\delta_{\mu_{E^1},\nu_{E^1}}. \tag{34}$$

The Hilbert space $\mathcal{H}_{\text{kin}}$ is the Cauchy completion of Cyl or more succinctly $\mathcal{H}_{\text{kin}} = L_2(\bar{\mathbb{R}}^3_{\text{Bohr}}, d\mu_0)$, where $\bar{\mathbb{R}}_{\text{Bohr}}$ is the Bohr-compactification of $\mathbb{R}$ and $d\mu_0$ is the Haar measure on $\bar{\mathbb{R}}^3_{\text{Bohr}}$.

We can quantize the theory using the standard quantization procedure $\{,\} \to -\frac{i}{\hbar}[,]$. We recall the fundamental difference from the standard Schrödinger quantization program. In the ordinary Schrödinger representation of the Weyl–Heisenberg algebra the classical fields c, A and A_1 translate in operators. In loop quantum gravity on the contrary the operators c, A and A_1 do not exist [32, 33]. We can not promote the Poisson brackets to commutators $[\hat{c}, \hat{p}] = i\gamma l_P^2/3$, $[\hat{A}, \hat{E}] = i\gamma l_P/4\pi$ and $[\hat{A}_1, \hat{E}^1] = i\gamma l_P/16\pi$; rather, the quantum fundamental operators are $\hat{c}, \hat{E}, \hat{E}^1, \hat{h}_I$. The momentum operators can be represented on the Hilbert space by

$$\hat{p} \to -i\frac{\gamma l_P^2}{3}\frac{d}{dc}, \qquad \hat{E} \to -i\frac{\gamma l_P}{4\pi}\frac{d}{dA}, \qquad \hat{E}^1 \to -i\frac{\gamma l_P}{16\pi}\frac{d}{dA_1}. \tag{35}$$

It is easy to calculate the spectrum of these two momentum operators on the Hilbert space basis. This is given by

$$\hat{p}|\mu, \mu_E, \mu_{E^1}\rangle = \frac{\mu\gamma l_P^2}{6}|\mu, \mu_E, \mu_{E^1}\rangle,$$

$$\hat{E}|\mu, \mu_E, \mu_{E^1}\rangle = \frac{\mu_E \gamma l_P^2}{8\pi}|\mu, \mu_E, \mu_{E^1}\rangle,$$
$$\hat{E}^1|\mu, \mu_E, \mu_{E^1}\rangle = \frac{\mu_{E^1}\gamma l_P}{16\pi\sqrt{2}}|\mu, \mu_E, \mu_{E^1}\rangle. \tag{36}$$

We have to fix the residual gauge freedom on the Hilbert space outside the matter. We consider the operator $\hat{P} : |\mu, \mu_E, \mu_{E^1}\rangle \to |\mu, \mu_E, -\mu_{E^1}\rangle$ and we impose that only the invariant states (under $\hat{P}$) are in the kinematical Hilbert space. The states in the Hilbert space are: $\frac{1}{\sqrt{2}}[|\mu, \mu_E, \mu_{E^1}\rangle + |\mu, \mu_E, -\mu_{E^1}\rangle]$ for $\mu_{E^1} \neq 0$ and the states $|\mu, \mu_E, 0\rangle$ for $\mu_{E^1} = 0$.

The holonomy operators $\hat{h}_I$ in the directions r, θ, ϕ of the space section $\mathbb{R} \times S^2$ outside the matter are: $\hat{h}_1^{(\mu_E)}$, $\hat{h}_2^{(\mu_{E^1})}$ and $\hat{h}_3^{(\mu_{E^1})}$, where $\mu_E l_P$ is the length along the radial direction r and μ_{E_1} is the length along the directions θ and ϕ (all the length are define using the fiducial triad ${}^o e_I^a$). Inside the matter the holonomy operators are $\hat{h}_I^{(RW)(\mu)}$where μ is the length along any direction χ, θ or ϕ inside the matter. The holonomies operators act on the Hilbert space $\mathcal{H}_{\text{kin}}$ by multiplication.

5 Hamiltonian Constraint and Quantum Dynamics

In this section we are going to study the dynamics of the model. The general strategy to quantize a system with constraints was introduced by Dirac. To implement the Dirac strategy we must impose the classical constraints at the quantum level to obtain the physical space. This strategy define the quantum Einstein's equations that in general are define by (for $\gamma = i$)

$$\hat{\mathcal{H}}|\psi\rangle = [-4e^{-1}\,\text{Tr}(F_{ab}E^a E^b) + \mathcal{H}_M]|\psi\rangle = 0,$$
$$\hat{\mathcal{H}}_b|\psi\rangle = [E_j^a F_{ab}^j + \mathcal{H}_{Mb}]|\psi\rangle = 0, \tag{37}$$
$$\hat{\mathcal{G}}_i|\psi\rangle = [\partial_a E_i^a + \epsilon_{ij}^k A_a^j E_k^a + \mathcal{G}_{Mi}]|\psi\rangle = 0.$$

In the case of our minisuperspace (homogeneous) model the second and the third constraints are identically zero then to obtain the physical space we have to solve the first scalar constraint. In this paper we want to quantize the model following the full loop quantum gravity ideas. To this aim we have expressed the Hamiltonian constraint in terms of homogeneous holonomies and we have introduced the polymer representation of the Weyl algebra. At this point we have all the ingredients to study the Hamiltonian constraint inside and outside the region where the matter is localized and to impose at quantum level the boundary condition introduced in Sect. 3.4.

As in non-trivially constrained systems, we expect that the physical states are not normalizable in the kinematical Hilbert space. However, as in the full loop quantum gravity theory, we again have the triplet

$$\text{Cyl} \subset \mathcal{H}_{\text{kin}} \subset \text{Cyl}^* \tag{38}$$

of spaces and the physical states will be in Cyl*, which is the algebraic dual of Cyl [1–5]. A generic element of this space is

$$\langle\psi| = \sum_{\mu, \mu_E, \mu_{E^1}} \psi_{\mu, \mu_E, \mu_{E^1}} \langle\mu, \mu_E, \mu_{E^1}|. \tag{39}$$

The quantum version of the Hamiltonian constraint outside the matter can be obtained promoting the classical holonomies to operators and the Poisson brackets to the commutators. Using the relations in (25) we can express the quantum version of the Kantowski–Sachs Hamiltonian constraint (26) as

$$\begin{aligned}\hat{H}_E = -\frac{16\pi i}{\mu_0^3 \gamma l_P^2}&[4\sin(2x)\sin(2y)(\sin(y)\hat{V}\cos(y) - \cos(y)\hat{V}\sin(y))\\&+ 2(\cos(\delta)\sin^2(2y) - \sin(\delta)(\sin^2(2y) + 2\cos(2y)))\\&\times(\sin(x)\hat{V}\cos(x) - \cos(x)\hat{V}\sin(x))],\end{aligned} \tag{40}$$

where we have introduced the following notations: $x = A\mu_0 l_P/2$, $y = \sqrt{2}A_1\mu_0/2$ and $\delta = \mu_0^2/2$.

Using the exponential form for the trigonometric function we can calculate the action of the Hamiltonian constraint on the Hilbert space basis (33) [14, 15]. At this point we can impose the Hamiltonian constraint outside the matter to obtain a first relation for the coefficients $\psi_{\mu,\mu_E,\mu_{E^1}}$ in the (39). The constraint equation $\hat{H}_E|\psi\rangle = 0$ is now interpreted as an equation in the dual space $\langle\psi|\hat{H}_E^\dagger$; from this equation we can derive a relation for the coefficients $\psi_{\mu,\mu_E,\mu_{E^1}}$

$$\begin{aligned}&(V_{\mu_E-2\mu_0,\mu_{E^1}-3\mu_0} - V_{\mu_E-2\mu_0,\mu_{E^1}-\mu_0})\psi_{\mu,\mu_E-2\mu_0,\mu_{E^1}-2\mu_0}\\&\quad+ (V_{\mu_E+2\mu_0,\mu_{E^1}-\mu_0} - V_{\mu_E+2\mu_0,\mu_{E^1}-3\mu_0})\psi_{\mu,\mu_E+2\mu_0,\mu_{E^1}-2\mu_0}\\&\quad+ (V_{\mu_E-2\mu_0,\mu_{E^1}+3\mu_0} - V_{\mu_E-2\mu_0,\mu_{E^1}+\mu_0})\psi_{\mu,\mu_E-2\mu_0,\mu_{E^1}+2\mu_0}\\&\quad+ (V_{\mu_E+2\mu_0,\mu_{E^1}+\mu_0} - V_{\mu_E+2\mu_0,\mu_{E^1}+3\mu_0})\psi_{\mu,\mu_E+2\mu_0,\mu_{E^1}+2\mu_0}\\&\quad+ \frac{\sin(\mu_0^2/2) - \cos(\mu_0^2/2)}{2}[(V_{\mu_E+\mu_0,\mu_{E^1}-4\mu_0} - V_{\mu_E-\mu_0,\mu_{E^1}-4\mu_0})\psi_{\mu,\mu_E,\mu_{E^1}-4\mu_0}\\&\quad- 2(V_{\mu_E+\mu_0,\mu_{E^1}} - V_{\mu_E-\mu_0,\mu_{E^1}})\psi_{\mu,\mu_E,\mu_{E^1}}\\&\quad+ (V_{\mu_E+\mu_0,\mu_{E^1}+4\mu_0} - V_{\mu_E-\mu_0,\mu_{E^1}+4\mu_0})\psi_{\mu,\mu_E,\mu_{E^1}+4\mu_0}]\\&\quad- 2\sin(\mu_0^2/2)[(V_{\mu_E+\mu_0,\mu_{E^1}-2\mu_0} - V_{\mu_E-\mu_0,\mu_{E^1}-2\mu_0})\psi_{\mu,\mu_E,\mu_{E^1}-2\mu_0}\\&\quad+ (V_{\mu_E+\mu_0,\mu_{E^1}+2\mu_0} - V_{\mu_E-\mu_0,\mu_{E^1}+2\mu_0})\psi_{\mu,\mu_E,\mu_{E^1}+2\mu_0}] = 0,\end{aligned} \tag{41}$$

where the volume eigenvalues are defined by

$$\begin{aligned}&\hat{V}|\mu,\mu_E,\mu_{E^1}\rangle = V_{\mu_E,\mu_{E^1}}|\mu,\mu_E,\mu_{E^1}\rangle,\\&V_{\mu_E,\mu_{E^1}} \equiv \frac{\gamma^{\frac{3}{2}} l_P^3}{4\sqrt{8\pi}}\sqrt{|\mu_E|}|\mu_{E^1}|.\end{aligned} \tag{42}$$

The other constraint that we must to impose on the state (39) is the Hamiltonian constraint inside the matter. If we introduce the holonomies (18) in (19) we obtain the following trigonometric form of the operator

$$\begin{aligned}\hat{H}_E^{(RW)} = &\frac{6(1-i)}{\mathrm{V}(\chi_0) l_P^2 \gamma \mu_0^3 \kappa} e^{-i\frac{c\mu_0(\mu_0+4)}{2}}\\&\times[-i + 2(1+i)e^{ic\mu_0} + 2ie^{2ic\mu_0} + 2(1+i)e^{3ic\mu_0} - ie^{4ic\mu_0}\end{aligned}$$

$$
\begin{aligned}
&+ e^{ic\mu_0^2} - 2(1+i)e^{ic\mu_0(\mu_0+1)} \\
&- 2e^{ic\mu_0(\mu_0+2)} - 2(1+i)e^{ic\mu_0(\mu_0+3)} + e^{ic\mu_0(\mu_0+4)}] \\
&\times \left[\sin\left(\frac{c\mu_0}{2}\right)\hat{V}^{(RW)}\cos\left(\frac{c\mu_0}{2}\right) - \cos\left(\frac{c\mu_0}{2}\right)\hat{V}^{(RW)}\sin\left(\frac{c\mu_0}{2}\right)\right], \qquad (43)
\end{aligned}
$$

(the operator c is not defined but the exponential and trigonometric operators are defined on the Hilbert space). We have simplified the notation using the identification $\delta_0 = \mu_0$.

The action of the operator (43) on the basis (33) is

$$
\begin{aligned}
&\hat{H}_E^{(RW)}|\mu, \mu_E, \mu_{E^1}\rangle \\
&= \frac{3(1+i)}{\gamma l_P^2 \kappa \mu_0^3}(V_{\mu+\mu_0} - V_{\mu-\mu_0})[-i|\mu - \mu_0(\mu_0+4), \mu_E, \mu_{E^1}\rangle \\
&\quad + 2(1+i)|\mu - \mu_0(\mu_0+2), \mu_E, \mu_{E^1}\rangle + 2i|\mu - \mu_0^2, \mu_E, \mu_{E^1}\rangle \\
&\quad + 2(1+i)|\mu - \mu_0(\mu_0-2), \mu_E, \mu_{E^1}\rangle - i|\mu - \mu_0(\mu_0-4), \mu_E, \mu_{E^1}\rangle \\
&\quad + |\mu + \mu_0(\mu_0-4), \mu_E, \mu_{E^1}\rangle - 2(1+i)\mu + \mu_0(\mu_0-2), \mu_E, \mu_{E^1}\rangle \\
&\quad - 2|\mu + \mu_0^2, \mu_E, \mu_{E^1}\rangle - 2(1+i)|\mu + \mu_0(\mu_0+2), \mu_E, \mu_{E^1}\rangle \\
&\quad + |\mu - \mu_0(\mu_0+4), \mu_E, \mu_{E^1}\rangle]. \qquad (44)
\end{aligned}
$$

We have all the ingredients to calculate the action of the Hamiltonian constraint inside the matter on the state (39). From equation $\langle\psi|(\hat{H}_E^{(RW)} + \hat{H}_M) = 0$ ($\langle\psi|\hat{H}_M = \langle\psi|M_{\text{dust}}, \forall\mu \neq 0$ and $\langle\psi|\hat{H}_M = 0$ for $m = 0$ [39]) we obtain another recursive relation for the coefficients $\psi_{\mu,\mu_E,\mu_{E^1}}$

$$
\begin{aligned}
&-i(V_{\mu+\mu_0(\mu_0+5)} - V_{\mu+\mu_0(\mu_0+3)})\psi_{\mu+\mu_0(\mu_0+4),\mu_E,\mu_{E^1}} \\
&\quad + 2(1+i)(V_{\mu+\mu_0(\mu_0+3)} - V_{\mu+\mu_0(\mu_0-1)})\psi_{\mu+\mu_0(\mu_0+2),\mu_E,\mu_{E^1}} \\
&\quad + 2i(V_{\mu+\mu_0(\mu_0+1)} - V_{\mu+\mu_0(\mu_0-1)})\psi_{\mu+\mu_0^2,\mu_E,\mu_{E^1}} \\
&\quad + 2(1+i)(V_{\mu+\mu_0(\mu_0-2)} - V_{\mu+\mu_0(\mu_0-3)})\psi_{\mu+\mu_0(\mu_0-2),\mu_E,\mu_{E^1}} \\
&\quad - i(V_{\mu+\mu_0(\mu_0-3)} - V_{\mu+\mu_0(\mu_0-4)})\psi_{\mu+\mu_0(\mu_0-4),\mu_E,\mu_{E^1}} \\
&\quad + (V_{\mu-\mu_0(\mu_0-5)} - V_{\mu-\mu_0(\mu_0-3)})\psi_{\mu-\mu_0(\mu_0-4),\mu_E,\mu_{E^1}} \\
&\quad - 2(1+i)(V_{\mu-\mu_0(\mu_0-3)} - V_{\mu-\mu_0(\mu_0-1)})\psi_{\mu-\mu_0(\mu_0-2),\mu_E,\mu_{E^1}} \\
&\quad - 2(V_{\mu-\mu_0(\mu_0-1)} - V_{\mu-\mu_0(\mu_0+1)})\psi_{\mu-\mu_0^2,\mu_E,\mu_{E^1}} \\
&\quad - 2(1+i)(V_{\mu-\mu_0(\mu_0+1)} - V_{\mu-\mu_0(\mu_0+3)})\psi_{\mu-\mu_0(\mu_0+2),\mu_E,\mu_{E^1}} \\
&\quad + (V_{\mu-\mu_0(\mu_0+3)} - V_{\mu-\mu_0(\mu_0+5)})\psi_{\mu-\mu_0(\mu_0+4),\mu_E,\mu_{E^1}} \\
&\quad + \frac{\gamma l_P^2 \kappa \mathrm{V}(\chi_0)\mu_0^3}{3(i+1)} M_{\text{dust}}\psi_{\mu,\mu_E,\mu_{E^1}} = 0, \qquad (45)
\end{aligned}
$$

where we have introduced the volume eigenvalue inside the matter

$$\hat{V}^{(RW)}|\mu,\mu_E,\mu_{E^1}\rangle = V_\mu|\mu,\mu_E,\mu_{E^1}\rangle,$$
$$V_\mu \equiv \mathrm{V}(\chi_0)\left(\frac{\gamma l_P^2}{6}\right)^{\frac{3}{2}}|\mu|^{\frac{3}{2}}. \tag{46}$$

The other constraints to impose are the boundary conditions in (31). The first of the two conditions assumes the following operator form

$$\frac{\sqrt{|p|}}{\mu_0}\left[4\sin\left(\frac{c\mu_0}{2}\right)+\mu_0\right]\sin(\chi_0) - 2\frac{\sqrt{2|E|}}{\mu_0}\sin\left(\frac{A_1\mu_0}{\sqrt{2}}\right) = 0, \tag{47}$$

and the action of the operator (47) on the state (39) impose another condition on the physical state for the gravitational collapse

$$\left(\sqrt{\frac{|\mu-\mu_0|}{3}}\psi_{\mu-\mu_0,\mu_E,\mu_{E^1}} - \sqrt{\frac{|\mu+\mu_0|}{3}}\psi_{\mu+\mu_0,\mu_E,\mu_{E^1}} - i\mu_0\sqrt{\frac{|\mu|}{3}}\psi_{\mu,\mu_E,\mu_{E^1}}\right)\sin^2(\chi_0)$$
$$- \sqrt{\frac{|\mu_E|}{2\pi}}(\psi_{\mu,\mu_E,\mu_{E^1}-\mu_0} - \psi_{\mu,\mu_E,\mu_{E^1}+\mu_0}) = 0. \tag{48}$$

The last constraint to impose is the second of equations (31). When we apply this constraint on the candidate physical state (39) we obtain

$$\sum_{\mu,\mu_E,\mu_{E^1}}[\sqrt{|p|}\sin(\chi_0) - \sqrt{|E|}]\psi_{\mu,\mu_E,\mu_{E^1}}|\mu,\mu_E,\mu_{E^1}\rangle$$
$$= \sum_{\mu,\mu_E,\mu_{E^1}}\left[\sqrt{\frac{|\mu|}{3}}\sin(\chi_0) - \sqrt{\frac{|\mu_E|}{4\pi}}\right]\psi_{\mu,\mu_E,\mu_{E^1}}|\mu,\mu_E,\mu_{E^1}\rangle = 0. \tag{49}$$

From (49) we obtain that the gravitational collapse wave function $\psi_{\mu,\mu_E,\mu_{E^1}}$ depends only on the two parameters μ and μ_{E^1}. For $\sin^2(\chi_0) = 3/4\pi$ we obtain

$$\psi_{\mu,\mu_E,\mu_{E^1}} \equiv \psi_{\mu,\mu,\mu_{E^1}}. \tag{50}$$

We conclude this section summarizing the quantum dynamical results. In this section we have imposed the quantum Einstein equations and we have obtained that the gravitational collapse wave function must satisfy the difference recursive equations (41), (45), (48) and then we must impose in the result the constraint (50). An important result in our simplified analysis is that all the coefficients in the difference equations (41), (45) and (48) are regular in $\mu = 0$ and in $\mu_E = 0$ where the classical Schwarzschild singularity is localized. We can conclude that we have a regular and natural evolution beyond the classical singular point. In the next paragraph we will show the consistency of the difference equations obtained in this section.

5.1 Physical States

In the previous paragraph we have calculated four difference equations (41), (45), (48) and (50) that the wave function for the gravitational collapse must satisfy. Now we analyze this system introducing the boundary conditions in the dynamics. We write the physical state

$|\psi\rangle$ in a tensor product form for the Region 1 and Region 2

$$\langle\psi| = \sum_{\mu} \varphi_{\mu}\langle\mu| \otimes \sum_{\mu_E,\mu_{E^1}} \phi_{\mu_E,\mu_{E^1}}\langle\mu_E,\mu_{E^1}|. \tag{51}$$

At this point to simplify the problem we follow the following steps.

1. The first difference equation is (45), this equation gives a relation for the coefficients φ_μ introduced in (51); we recall (45) using the simplification $\mu_0 = 1$ and introducing integer values for the independent variable μ, $\mu = m\mu_0 \equiv m$ ($m \in 2\mathbb{Z}$ [2])

$$\begin{aligned}
&-i(V_{m+6} - V_{m+4})\varphi_{m+5} + 2(1+i)(V_{m+4} - V_m)\varphi_{m+3} + 2i(V_{m+2} - V_m)\varphi_{m+1}\\
&\quad + 2(1+i)(V_{m-1} - V_{m-2})\varphi_{m-1} - i(V_{m-2} - V_{m-3})\varphi_{m-3} + (V_{m+4} - V_{m+2})\varphi_{m+3}\\
&\quad - 2(1+i)(V_{m+2} - V_m)\varphi_{m+1} - 2(V_m - V_{m-2})\varphi_{m-1} - 2(1+i)(V_{m-2} - V_{m-4})\varphi_{m-3}\\
&\quad + (V_{m-4} - V_{m-6})\psi_{m-5} + \frac{\gamma l_P^2 \kappa \mathrm{V}(\chi_0)}{3(i+1)} M_{\mathrm{dust}}\varphi_m = 0,
\end{aligned} \tag{52}$$

this difference equation can be solved for φ_{m_0+5} introducing the following initial conditions: $\varphi_{m_0-5}, \varphi_{m_0-3}, \varphi_{m_0-1}, \varphi_{m_0+1}$ and φ_{m_0+3} ($m_0 \in 2\mathbb{Z}$). The component φ_{m_0} that is the even component in (52) can be calculated using the constraint (48) with $\mu_E = 0$ as we will stress at the point 3. of this paragraph.

2. The second equation is (41) with $\mu_0 = 1$, $\mu_{E^1} = 2n$ and $\mu_E = 2l$ with $n, l \in \mathbb{Z}$

$$\begin{aligned}
&(V_{2l-2,2n-3} - V_{2l-2,2n-1})\phi_{2l-2,2n-2} + (V_{2l+2,2n-1} - V_{2l+2,2n-3})\phi_{2l+2,2n-2}\\
&\quad + (V_{2l-2,2n+3} - V_{2l-2,2n+1})\phi_{2l-2,2n+2} + (V_{2l+2,2n+1} - V_{2l+2,2n+3})\phi_{2l+2,2n+2}\\
&\quad + (\sin(\mu_0^2/2) - \cos(\mu_0^2/2))[(V_{2l+1,2n-4} - V_{2l-1,2n-4})\phi_{2l,2n-4}\\
&\quad - 2(V_{2l+1,2n} - V_{2l-1,2n})\phi_{2l,2n} + (V_{2l+1,2n+4} - V_{2l-1,2n+4})\phi_{2l,2n+4}]/2\\
&\quad - 2\sin(\mu_0^2/2)[(V_{\mu_E+\mu_0,\mu_{E^1}-2\mu_0} - V_{2l-1,2n-2})\phi_{2l,2n-2}\\
&\quad + (V_{2l+1,2n+2} - V_{2l-1,2n+2})\phi_{2l,2n+2}] = 0.
\end{aligned} \tag{53}$$

We can solve this difference equation to obtain $\phi_{2l,2n}$ giving the boundary conditions: $\phi_{-2l_0,2n}, \phi_{2l_0,2n}$ $\forall n$ and $\phi_{2l,2n_0-4}, \phi_{2l,2n_0-2}, \phi_{2l,2n_0}, \phi_{2l,2n_0+2}$ $\forall l$.

3. The last equation is (48) with $\sin^2(\chi_0) = 3/4\pi$ (this is useful to simplify the notation), in this equation we introduce $\mu_E = 2l$ and $\mu_{E^1} = 2n$ as in the previous equation

$$\begin{aligned}
&(3/4\pi)(\sqrt{|m-1|}\varphi_{m-1} - \sqrt{|m+1|}\varphi_{m+1} - i\sqrt{|m|}\varphi_m)\phi_{2l,2n}\\
&\quad - \sqrt{3|l|/\pi}\varphi_m(\phi_{2l,2n-1} - \phi_{2l,2n+1}) = 0.
\end{aligned} \tag{54}$$

(a) For $l = 0$ we obtain a relation to express φ_m in terms of φ_{m+1} and φ_{m-1}. In particular introducing the initial conditions φ_{m_0-1} and φ_{m_0+1} (that are the same initial condition introduced in (52)) we can obtain φ_{m_0} from equation

$$\sqrt{|m_0-1|}\varphi_{m_0-1} - \sqrt{|m_0+1|}\varphi_{m_0+1} - i\sqrt{|m_0|}\varphi_{m_0} = 0. \tag{55}$$

[2] It is possible repeat the same analysis for the components with $m \in 2\mathbb{Z}+1$.

At this point we have the even component φ_{m_0} useful to solve at the same time equation (52) and (55) (see point 1).
We observe that if we introduce $m=0$ (and obviously $l=0$) in equation (54) we obtain $\varphi_{-1}=\varphi_1$, φ_0 decouples from the difference equation; this means that not all the initial boundary conditions $\varphi_{m_0-5}, \varphi_{m_0-3}, \varphi_{m_0-1}, \varphi_{m_0+1}, \varphi_{m_0+3}$ are independent; there is one consistency relation between the five initial conditions.

(b) For $l \neq 0$ we insert the solution φ_m of (52) in (54) and we obtain a second order difference equation for $\phi_{2l,2n}$ in the variable n. Using the constraint (50) for $\sin^2(\chi_0) = 3/4\pi$ we can evaluate equation (54) on $m=\mu_E=2l$ to obtain

$$\begin{aligned}&(3/4\pi)(\sqrt{|m-1|}\varphi_{m-1} - \sqrt{|m+1|}\varphi_{m+1} - i\sqrt{|m|}\varphi_m)|_{m=2l}\phi_{2l,2n}\\ &\quad - \sqrt{3|l|/\pi}\,\varphi_m|_{m=2l}(\phi_{2l,2n-1} - \phi_{2l,2n+1}) = 0. \end{aligned} \tag{56}$$

Introducing the initial condition $\phi_{2l,2n_0-1}\ \forall l$ and $\phi_{2l,2n_0}\ \forall l$ (this second condition can be obtained from equation (53)), we can calculate $\phi_{2l,2n_0+1}$. This observation implies that in the variable "n", from equation (53) we can calculate the even components $\phi_{2l,2n}$ and from (56) we calculate the odd components $\phi_{2l,2n+1}$ of the wave function.

We conclude the section summarizing the results about physical states. Using the news notations we can write the physical states in the following way

$$\langle\psi| = \sum_m \varphi_m\langle m| \otimes \sum_{l,n}\phi_{2l,n}\langle 2l,n|, \tag{57}$$

where φ_m is calculated using the two difference equations (52) and (55), and $\phi_{2l,n}$ can be obtained from equations (53) and (56).

6 Conclusions

In this paper we have studied the gravitational collapse in Ashtekar variables following the paper [18]. We have studied the gravitational collapse when all dust matter has crossed the event horizon. In this particular region the metric is homogeneous and we have applied the technology developed in "loop quantum cosmology" and in "loop quantum black hole" papers [12, 14, 15, 31]. We have divided the space-time region inside the event horizon in two parts, the firs one where the matter is localized and the other one part outside the mater. The space-time metric inside the matter is of Friedmann–Robertson–Walker type and the metric outside the matter is of Kantowski–Sachs type with spatial topology $\mathbf{R}\times\mathbf{S}^2$.

The quantization procedure is induced by full "loop quantum gravity". We have introduced homogeneous holonomies and we have expressed the Hamiltonian constraint in terms of such holonomies in all the region inside the event horizon. The main result is that the Hamiltonian constraint gives a regular difference equation for the coefficients of the physical states which are defined in the dual space of the dense subspace of the kinematical Hilbert space. We can summarize this result recalling that the quantum Einstein's equations

$$\begin{aligned}&(\psi|\hat{H}_E(h,V) = 0,\\ &(\psi|(\hat{H}_E^{RW}(h^{RW},V^{RW}) + \hat{H}_M) = 0,\end{aligned} \tag{58}$$

and the boundary conditions are regular in $r=0$ where the classical singularity is localized (h, V and h^{RW}, V^{RW} are respectively the holonomy and the volume inside and outside the matter).

An important consequence of the quantization is that, unlike the classical evolution, the quantum evolution does not stop at the classical singularity and the "other side" of the singularity corresponds with a new domain where the triad reverses its orientation. In this simply model we have imposed the quantum Einstein's equations obtaining recursive consistent equations for the coefficients $\psi_{\mu,\mu_E,\mu_{E^1}}$ of the physical states

$$\langle\psi| = \sum_{\mu,\mu_E,\mu_{E^1}} \psi_{\mu,\mu_E,\mu_{E^1}} \langle\mu, \mu_E, \mu_{E^1}|. \tag{59}$$

We can interpret $\psi_{\mu,\mu_E,\mu_{E^1}}$ as the wave function of the gravitational collapse, after the matter has crossed the event horizon. We have also showed the consistency of the four difference equations. In a future paper we will study the difference equations (41), (45), (48) to obtain a numerical solution of those equations for a particular value of the boundary condition on the wave function.

We hope that the analysis performed here will shed light on the problem of the "information loss" in the process of black hole formation and evaporation. See in particular [41] for a possible physical interpretation of the black hole information loss problem.

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